Support Document for **Order-2 Probabilistic Information Fusion on Random Permutation Set**

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1 Appendix 1: Algorithm 1

Algorithm 1 Order-2 structural correction. $Prob_{OTP}^{\mathcal{Y}} = O2SC\left(Prob_{TP}^{\mathcal{Y}}, \{Prob_1^{\mathcal{Y}}, \cdots, Prob_N^{\mathcal{Y}}\}, \lambda, \boldsymbol{M}_{\mathrm{D}}\right)$

Input: probability distributions of sources $Prob_1^{\mathcal{Y}}, \cdots, Prob_N^{\mathcal{Y}}$; traditional outcome $Prob_{TP}^{\mathcal{Y}}$; dependency matrix $\boldsymbol{M}_{\mathrm{D}}$; entanglement degree λ;

Output: outcome on order-2 structure $Prob_{OTP}^{\mathcal{Y}}$;

% *Generate maximum structural entanglement belief* $\boldsymbol{E}$.

for $i \leftarrow 1 : |\mathcal{Y}|$ **do**
  $E(A_i^{\mathcal{X}}) \leftarrow 1 - \frac{M_{\mathrm{D}}(i,i)}{\sum M_{\mathrm{D}}(:,i)}$;
end for

% *Construct entanglement matrix* $\boldsymbol{M}_{\mathrm{E}}$.

for $i \leftarrow 1 : |\mathcal{Y}|$ **do**
  for $j \leftarrow 1 : |\mathcal{Y}|$ **do**
    if $i = j$ **then**
      $M_{\mathrm{E}}(i,i) \leftarrow 1 - E(A_i^{\mathcal{X}}) \times \lambda$;
    else
      $M_{\mathrm{E}}(i,j) \leftarrow \frac{E(A_j^{\mathcal{X}}) \times \lambda}{\sum M_{\mathrm{D}}(:,j) - M_{\mathrm{D}}(j,j)}$
    end if
  end for
end for

% *Correct belief updating.*

for $i \leftarrow 1 : N$ **do**
  $\boldsymbol{\Delta}_i \leftarrow \boldsymbol{M}_{\mathrm{E}} \times \left(\boldsymbol{Prob}_{TP}^{\mathcal{Y}} - \boldsymbol{Prob}_i^{\mathcal{Y}}\right)$;
  $\boldsymbol{Prob}_{\mathrm{C,i}}^{\mathcal{Y}} \leftarrow \boldsymbol{Prob}_i^{\mathcal{Y}} + \boldsymbol{\Delta}_i$;
end for
$\boldsymbol{Prob}_{\mathrm{temp}}^{\mathcal{Y}} \leftarrow \frac{1}{N} \sum_{i=1}^{N} \boldsymbol{Prob}_{\mathrm{C,i}}^{\mathcal{Y}}$;

% *Normalize the probability distribution.*

if $\min \boldsymbol{Prob}_{\mathrm{temp}}^{\mathcal{Y}} < 0$ **then**
  $Prob_{OTP}^{\mathcal{Y}}(A_i^{\mathcal{X}}) \leftarrow \frac{Prob_{\mathrm{temp}}^{\mathcal{Y}}(A_i^{\mathcal{X}}) + \min \boldsymbol{Prob}_{\mathrm{temp}}^{\mathcal{Y}}}{\sum\left(\boldsymbol{Prob}_{\mathrm{temp}}^{\mathcal{Y}} + \min \boldsymbol{Prob}_{\mathrm{temp}}^{\mathcal{Y}}\right)}$;
else
  $\boldsymbol{Prob}_{OTP}^{\mathcal{Y}} \leftarrow \boldsymbol{Prob}_{\mathrm{temp}}^{\mathcal{Y}}$;
end if

Table 1: Relationships of notions in the aforementioned uncertainty models.

Model	event space	order-2 proposition	order-1 proposition	information distribution
order-2 information granule	reference information granules $\mathcal{X}$,$\mathcal{Y}$,$\cdots$	reference information granule A_i	mutually exclusive element x_i	probability distribution $Prob^{\mathcal{X}}$
Dempster -Shafer theory	frame Ω	focal set F_i	element ω_i	basic probability assignment m
layer-2 belief structure	frame Ω	ordered focal set F_i^j	element ω_i	permutation mass function $Perm$
multi-object recognition	measurable space $\mathcal{D}$	multi-object $\{d_1 d_2 \cdots\}$	object d_i	probability distribution $Prob^{\mathcal{D}}$
random finite set	power set 2^{Ω}	multi-element proposition F_i	element ω_i	probability distribution $Prob^{2^{\Omega}}$
random permutation set	permutation event space $PES(\Omega)$	ordered multi-element proposition F_i^j	element ω_i	probability distribution $Prob^{PES(\Omega)}$

2 Appendix 2: Table 1

3 Appendix 3: Example 4.

Example 1. *Consider two order-2 probability distributions* $Prob_1^{\mathcal{Y}} = \{0.8, 0.15, 0.05\}$ *and* $Prob_2^{\mathcal{Y}} = \{0.7, 0.25, 0.05\}$, *and the dependency matrix is* $\boldsymbol{M}_{\mathrm{D}} = \begin{bmatrix} 1 & 0 & 0.8 \\ 0 & 1 & 0 \\ 0.8 & 0 & 1 \end{bmatrix}$. *Suppose the contextual knowledge consists of* $\boldsymbol{R} = \{1, 1\}$, $\alpha = 0$, *and* $\lambda = 0.7$. *The calculation processes of SLF-CR and its order-2 structural version are as follows:*

1. *Generate ordered weights vector,* $\boldsymbol{w} = \{0, 1\}$.
2. *Calculate the SLF,*
$$\boldsymbol{L} = \{0.56, 0.0375, 0.0025\}.$$
3. *Calculate the outcome of SLF-CR,*
$$\boldsymbol{Prob}_{\oplus}^{\mathcal{Y}} = \{0.9333, 0.0625, 0.0042\}.$$
4. *Generate the entanglement matrix,*
$$\boldsymbol{M}_{\mathrm{E}} = \begin{bmatrix} 0.6889 & 0 & 0.3111 \\ 0 & 1 & 0 \\ 0.3111 & 0 & 0.6889 \end{bmatrix}.$$
5. *According to the Line 13-16 in Algorithm 1, the corrected distributions are*
$$\boldsymbol{Prob}_{\mathrm{C},1}^{\mathcal{Y}} = \{0.8776, 0.0625, 0.0599\},$$
$$\boldsymbol{Prob}_{\mathrm{C},2}^{\mathcal{Y}} = \{0.8465, 0.0625, 0.0910\}.$$
6. *According to the Line 17-21 in Algorithm 1, the outcome of order-2 SLF-CR is*
$$\boldsymbol{Prob}_{\oplus,\mathrm{O2}}^{\mathcal{Y}} = \{0.8620, 0.0625, 0.0755\}.$$

4 Appendix 3: probabilities in Example 8.

4.1 Probability distributions from sources

$$Prob_1(F_1) = 0.12, Prob_1(F_2) = 0.1, Prob_1(F_3^1) = 0.15,$$
$$Prob_1(F_3^2) = 0.15, Prob_1(F_5^1) = 0.03, Prob_1(F_7^1) = 0.1,$$
$$Prob_1(F_7^2) = 0.25, Prob_1(F_7^3) = 0.08, Prob_1(F_7^5) = 0.02,$$

$$Prob_2(F_1) = 0.1, Prob_2(F_2) = 0.15, Prob_2(F_3^1) = 0.07,$$
$$Prob_2(F_3^2) = 0.23, Prob_2(F_5^1) = 0.1, Prob_2(F_7^1) = 0.1,$$
$$Prob_2(F_7^2) = 0.05, Prob_2(F_7^3) = 0.12, Prob_2(F_7^5) = 0.08,$$

$$Prob_3(F_1) = 0.18, Prob_3(F_2) = 0.05, Prob_3(F_3^1) = 0.15,$$
$$Prob_3(F_3^2) = 0.05, Prob_3(F_5^1) = 0, Prob_3(F_7^1) = 0.12,$$
$$Prob_3(F_7^2) = 0.3, Prob_3(F_7^3) = 0.1, Prob_3(F_7^5) = 0.05.$$

4.2 Fusion outcomes

$$Prob_{\mathrm{D}}(F_1) = 0.177, Prob_{\mathrm{D}}(F_2) = 0.062, Prob_{\mathrm{D}}(F_3^1) = 0.129,$$
$$Prob_{\mathrm{D}}(F_3^2) = 0.141, Prob_{\mathrm{D}}(F_5^1) = 0, Prob_{\mathrm{D}}(F_7^1) = 0.098,$$
$$Prob_{\mathrm{D}}(F_7^2) = 0.307, Prob_{\mathrm{D}}(F_7^3) = 0.079, Prob_{\mathrm{D}}(F_7^5) = 0.007.$$

$$Prob_{\mathrm{L}}(F_1) = 0.35, Prob_{\mathrm{L}}(F_2) = 0.201, Prob_{\mathrm{L}}(F_3^1) = 0.251,$$
$$Prob_{\mathrm{L}}(F_3^2) = 0.063, Prob_{\mathrm{L}}(F_5^1) = 0.033, Prob_{\mathrm{L}}(F_5^2) = 0.003,$$
$$Prob_{\mathrm{L}}(F_7^1) = 0.021, Prob_{\mathrm{L}}(F_7^2) = 0.052, Prob_{\mathrm{L}}(F_7^3) = 0.017,$$
$$Prob_{\mathrm{L}}(F_7^5) = 0.09.$$

$$Prob_{\oplus}(F_1) = 0.135, Prob_{\oplus}(F_2) = 0.092, Prob_{\oplus}(F_3^1) = 0.121,$$
$$Prob_{\oplus}(F_3^2) = 0.137, Prob_{\oplus}(F_4) = 0.003, Prob_{\oplus}(F_5^1) = 0.048,$$
$$Prob_{\oplus}(F_5^2) = 0.006, Prob_{\oplus}(F_6^1) = 0, Prob_{\oplus}(F_6^2) = 0.006,$$
$$Prob_{\oplus}(F_7^1) = 0.104, Prob_{\oplus}(F_7^2) = 0.19, Prob_{\oplus}(F_7^3) = 0.096,$$
$$Prob_{\oplus}(F_7^4) = 0.004, Prob_{\oplus}(F_7^5) = 0.053, Prob_{\oplus}(F_7^6) = 0.005.$$